

BRANES THROUGH FINITE GROUP ACTIONS

SEBASTIAN HELLER & LAURA P. SCHAPOSNIK

ABSTRACT. Mid-dimensional (A, B, A) and (B, B, B) -branes in the moduli space of flat $G_{\mathbb{C}}$ -connections appearing from finite group actions on compact Riemann surfaces are studied. The geometry and topology of these spaces is then described via the corresponding Higgs bundles and Hitchin fibrations.

1. INTRODUCTION

This paper is dedicated to study mid-dimensional subspaces of the moduli space of flat $G_{\mathbb{C}}$ -connections on a compact Riemann surface Σ of genus $g \geq 2$, for $G_{\mathbb{C}}$ a complex Lie group, associated to finite group actions on Σ . As shown in [14, 28], considering suitable stability conditions, a Higgs bundle defines a solution of equations for a G -connection A known as the Hitchin equations, where G is the maximal compact subgroup of $G_{\mathbb{C}}$. In particular, for $G = U(n)$ these are $F_A + [\Phi, \Phi^*] = 0$ and then the connection $\nabla_A + \Phi + \Phi^*$ is flat, with holonomy in $GL(n, \mathbb{C})$. We shall denote by $\mathcal{M}_{G_{\mathbb{C}}}$ the moduli space of $G_{\mathbb{C}}$ -Higgs bundles on a compact Riemann surface Σ , the space of solutions to the Hitchin equations on the surface.

The smooth locus of $\mathcal{M}_{G_{\mathbb{C}}}$ is a hyperkähler manifold, and thus there is a family of complex structures from which we shall fix I, J, K obeying the same relations as the imaginary quaternions, following the notation of [5, 14, 18]. Along the paper we adopt the physicists' language in which a Lagrangian submanifold supporting a flat connection is called an *A-brane*, and a complex submanifold supporting a complex sheaf is a *B-brane*. A submanifold of a hyperkähler manifold may be of type *A* or *B* with respect to each of the structures and hence one may speak of branes of type (B, B, B) , (B, A, A) , (A, B, A) and (A, A, B) . After overviewing finite group actions on flat connections, in Section 2.2 natural (B, B, B) -branes are constructed:

Theorem 7: *Let Σ be a compact Riemann surface of genus $g \geq 2$ and Γ a finite group acting on Σ by holomorphic automorphisms. The connected components of the space of gauge equivalence classes of irreducible Γ -equivariant flat $G_{\mathbb{C}}$ -connections are hyper-Kähler submanifolds of the moduli space of flat irreducible $G_{\mathbb{C}}$ -connections on Σ , and hence give (B, B, B) -branes in the moduli space of $G_{\mathbb{C}}$ -Higgs bundles.*

Since mid-dimensional spaces may be *A*-branes or *B*-branes with respect to each of the structures, it is particularly natural to seek finite group actions giving mid-dimensional hyper-Kähler submanifolds. In Section 3 we give a classification of actions leading to mid-dimensional branes in the moduli space of flat $G_{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$ -connections.

Theorem 11: *Let Σ be a compact Riemann surface of genus $g \geq 2$ and Γ be a finite group acting on Σ by holomorphic automorphisms such that a component of the moduli space of Γ -equivariant flat $\mathrm{SL}(2, \mathbb{C})$ -connections on Σ has half the dimension of the moduli space \mathcal{M}_g . Then one of the following holds:*

- (I) $\Gamma = \mathbb{Z}_2$ acts by a fix-point free involution on Σ , or
- (II) Σ is hyperelliptic of genus 3 and $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$.

In the later case (II), one of the \mathbb{Z}_2 -factors corresponds to the hyperelliptic involution, whilst the other \mathbb{Z}_2 -factor corresponds to an involution with 4 fixed points.

Although we highlight results which hold for generic groups, most of the work in the remaining sections is done for $G_{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$. In order to understand the geometry of these branes, we consider them inside the moduli space of Higgs bundles and look at their intersection with smooth fibres of the Hitchin fibration. After overviewing the $\mathrm{SL}(2, \mathbb{C})$ -Hitchin fibration in Section 4, we obtain the following geometric description in Section 5 of the intersection of the (B, B, B) -brane of Theorem 11 (I) with the regular fibres of the Hitchin fibration:

Theorem 13: *Let τ be a fix point free involution. The τ -equivariant (B, B, B) -brane intersects a generic fibre of the corresponding Hitchin fibration over a point defining the spectral curve S in the abelian variety $\mathrm{Prym}(S/\tau, \Sigma/\tau)/\mathbb{Z}_2$.*

In order to study the equivariant (B, B, B) -branes of Theorem 11 (II), it is shown in Section 6 that it suffices to consider Higgs bundles over hyperelliptic surfaces of genus 3 surfaces with fixed point free actions, and describe the intersection of the brane with the regular fibres in this case in Theorem 24. Finally, from [18, Section 12], the equivariant and anti-equivariant spaces considered in this short paper have dual branes in the moduli spaces of Higgs bundles for the Langlands dual group. We conclude the work in Section 7 with comments on Langlands duality for the branes constructed in this paper. Since involutions in the moduli space of Higgs bundles have been considered in recent years in order to obtain families of branes, it is interesting to compare the spaces in Theorem 7 with spaces appearing in other papers - e.g. the real integrable systems given by (A, B, A) -branes in [4, Theorem 17] and the (B, A, A) -branes appearing in [5].

1.1. Acknowledgements. This paper begun during the workshop *Higgs Bundles in Geometry and Physics* at the University of Heidelberg Feb 28-March 3 2016, and the authors would like to thank the organizers for such a stimulating environment. The work of LPS is partially supported by NSF DMS-1509693 and the work of SH is partially supported by DFG HE 6828/1-2.

2. EQUIVARIANT FLAT CONNECTIONS

Consider Σ a compact (connected) Riemann surface of genus $g \geq 2$ and a finite group action $\Gamma \times \Sigma \rightarrow \Sigma$. These actions have been studied by many researchers and in the case of surfaces of genus 2 and 3, a complete classification of all finite group actions is given in [6, Table 5]. Moreover, in the case of actions induced on rank 2 bundles through automorphisms of Σ , a very concrete description of the fixed points in terms of parabolic structures is given in [1].

In order to understand flat equivariant $G_{\mathbb{C}}$ -connections on Σ , one needs to first fix a C^∞ trivialization $\underline{\mathbb{C}}^n = \Sigma \times \mathbb{C}^n \rightarrow \Sigma$ of the underlying vector bundle. In what follows we shall restrict our attention to the groups $G_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, and thus in the case of $\mathrm{SL}(n, \mathbb{C})$ require the trivialization to preserve the determinant.

Definition 1. *A Γ -equivariant flat connection on Σ is a flat connection ∇ on $\underline{\mathbb{C}}^n \rightarrow \Sigma$ such that for every $\psi \in \Gamma$ there exist a $\mathrm{GL}(n, \mathbb{C})$ -gauge transformation $g_\psi: \Sigma \rightarrow \mathrm{GL}(n, \mathbb{C})$ for which*

$$\psi^* \nabla = \nabla \cdot g_\psi,$$

and where $\psi \mapsto g_\psi$ is a generalized group homomorphism, this is, satisfies $g_{\mathrm{id}} = \mathrm{id}$ and $g_{(\psi \circ \tau)}(p) = g_\psi \circ g_\tau(\psi(p))$.

Denote by $\Gamma(\Sigma, G_{\mathbb{C}})$ the space of such $G_{\mathbb{C}}$ -gauge transformation on the Riemann surface Σ . Note that the stabilizer subgroup of the gauge action for an irreducible flat connection ∇ is contained in the diagonal constants, $\mathrm{Stab}_\nabla \subseteq \{g = \lambda \mathrm{id} \mid \lambda \in \mathbb{C}\}$, with equality in the case of $G_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$. In the case of $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$ one may not always be able to choose the generalized group homomorphism to be of the form

$$(1) \quad \psi \in \Gamma \mapsto g_\psi \in \Gamma(\Sigma, \mathrm{SL}(n, \mathbb{C})).$$

When (1) holds, the connections are Γ -equivariant flat $\mathrm{SL}(n, \mathbb{C})$ -connection on Σ . On the other hand, there are cases in which the generalized group homomorphism cannot be chosen to take values in the $\mathrm{SL}(n, \mathbb{C})$ -gauge group: an example of this is the so-called hyperelliptic descent of flat $\mathrm{SL}(2, \mathbb{C})$ -connections on a genus 2 surface considered in [13, §2]. As can be seen in such example, the dimensions of the two corresponding components, determined by whether the gauges are $\mathrm{SL}(2, \mathbb{C})$ -valued or not, might have different dimensions for the same group action.

2.1. Finite group actions on Riemann surfaces. Along the paper we shall distinguish between finite group actions on Riemann surfaces with or without fix points. In either case, Γ -equivariant flat $G_{\mathbb{C}}$ -connection can be studied through the quotient surface. Whilst these results might be classical, some proofs have been included here since sources of reference for them could not be found. We restrict to the case of $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C}), \mathrm{GL}(n, \mathbb{C})$:

Proposition 2. *Let $\Gamma \times \Sigma \rightarrow \Sigma$ be a finite group action by holomorphic automorphisms without fix points. Then, any Γ -equivariant flat $G_{\mathbb{C}}$ -connection is given by the pull-back of a flat $\mathrm{GL}(n, \mathbb{C})$ -connection on Σ/Γ .*

For $\Gamma \times \Sigma \rightarrow \Sigma$ a finite group action by holomorphic automorphisms with fix points, denote by $B \subset \Sigma/\Gamma$ the image of the fixed points, giving the branch points of the ramified cover

$$(2) \quad \pi_{\Gamma} : \Sigma \rightarrow \Sigma/\Gamma.$$

Proposition 3. *Let $\Gamma \times \Sigma \rightarrow \Sigma$ be a group action by holomorphic automorphisms with fixed points, and let B be the image of the fixed points in Σ/Γ . Then, any Γ -equivariant flat $G_{\mathbb{C}}$ -connection is given by the pull-back of a flat $\mathrm{GL}(n, \mathbb{C})$ -connection on $\Sigma/\Gamma - B$.*

Remark 4. *If $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$ and the generalized group homomorphism $\psi \mapsto g_{\psi}$ takes values in the $\mathrm{SL}(n, \mathbb{C})$ -gauge group, then in Propositions 2-3 the connection on Σ/Γ , resp. on $\Sigma/\Gamma - B$, can be chosen to be a $\mathrm{SL}(n, \mathbb{C})$ -connection. If the generalized group homomorphism $\psi \mapsto g_{\psi}$ does not take values in the $\mathrm{SL}(n, \mathbb{C})$ -gauge group, then it is more appropriate (e.g., for counting dimensions of the moduli spaces) to consider the induced $\mathrm{PSL}(n, \mathbb{C})$ -connection instead of the $\mathrm{GL}(n, \mathbb{C})$ -connection constructed in Propositions 2-3.*

The local monodromy of the equivariant connections around images $p \in B$ of branch points can be described by considering the stabilizer group of p , leading to the following:

Proposition 5. *Given a Γ -equivariant $G_{\mathbb{C}}$ -connection on Σ , the conjugacy class of the monodromy along a small loop around a point $p \in B$ is given by a root of the identity. Moreover, for sufficiently close irreducible flat Γ -equivariant connections, the conjugacy classes of these local monodromies are the same.*

Proof. Recall that the stabilizer group $\mathrm{Stab}_p \subset \Gamma$ of a point $p \in \Sigma$ satisfies $\mathrm{Stab}_p = \mathbb{Z}_k$ for some $k \geq 1$, and is non-trivial if and only if p is a branch point of $\pi_{\Gamma} : \Sigma \rightarrow \Sigma/\Gamma$. The local monodromy around $\pi_{\Gamma}(p) =: q$ can then be determined as follows: given ψ a generator of Stab_p , one has that $\psi^k = \mathrm{id}$ and $g_{\psi}(p)^k = \mathrm{id}$. Note that if $\tau(p) = \tilde{p} \neq p$ for some $\tau \in \Gamma$, then the order of the stabilizer group of \tilde{p} is the same as the one of p , and the conjugacy class of the corresponding gauge $g_{\tau \circ \psi \circ \tau^{-1}}(\tilde{p})$ as a subset of $\mathrm{GL}(n, \mathbb{C})$ is also the same as the conjugacy class of $g_{\psi}(p)$. Then, $g_{\psi}(p)$ represents the conjugacy class of the local monodromy around q , and this can be seen as follows. Indeed, consider a singular connection $\tilde{\nabla}$ on the punctured disc. By means of the Deligne extension procedure, the connection $\tilde{\nabla}$ is gauge equivalent to

$$(3) \quad d + A \frac{dz}{z}$$

for some $A \in \mathfrak{g}_{\mathbb{C}}$. Then, near $p \in \Sigma$ the connection $\nabla = d + \omega$ is gauge equivalent via a singular (but single valued) gauge transformation g to the pullback of (3) via the covering

map $z \mapsto z^k$, and this gauge transformation is invariant under $z \mapsto e^{\frac{2\pi i}{k}} z$. On the other hand, the transformation g can be written as $g = z^{-kA} h(z)$ where h is a single valued smooth map $U \subset \Sigma \rightarrow G_{\mathbb{C}}$. The conjugacy class of the local monodromy of (3) is given by z^{-A} which proves the claim as g is invariant and $\omega = h^{-1}dh$.

Finally, in order to see that nearby irreducible flat equivariant connections ∇^1 and ∇^2 give rise to the same local monodromies on the quotient surface, consider gauges g_{ψ}^i for which $\psi^* \nabla^i = \nabla^i \cdot g_{\psi}^i$ and $g_{\psi}^i(p)^k = \text{id}$. These gauges are parallel sections with respect to

$$(4) \quad \nabla^i \otimes (\psi^* \nabla^i)^*,$$

and as the connections are irreducible, the space of parallel sections is a 1-dimensional complex space. Hence $g^1(p)$ and $g^2(p)$ are close to each other if the connections ∇^1 and ∇^2 are close, and since both $g^1(p)$ and $g^2(p)$ are unipotent, they give rise to the same conjugacy class. \square

As mentioned in Section 1, subspaces of the smooth loci of the moduli space of $G_{\mathbb{C}}$ -Higgs bundles, or equivalently, $G_{\mathbb{C}}$ -flat connections, may be holomorphic or Lagrangian with respect to some of the fixed complex structures and symplectic forms, giving what we refer to as B -branes and A -branes in those structures [18].

2.2. Equivariant (B, B, B) -branes. In what follows we shall show that the moduli space of Γ -equivariant flat irreducible $G_{\mathbb{C}}$ -connections on a Riemann surface Σ is a well-defined hyper-Kähler submanifold of the moduli space of flat irreducible $G_{\mathbb{C}}$ -connections on Σ . From the previous sections, this space is a complex submanifold of the moduli space of irreducible flat $G_{\mathbb{C}}$ -connections on Σ (with respect to the structure J induced by the complex group $G_{\mathbb{C}}$).

Lemma 6. *Let ∇ be a Γ -equivariant flat irreducible $G_{\mathbb{C}}$ -connection on Σ , and $g: \Sigma \rightarrow G_{\mathbb{C}}$ be a gauge transformation. Then, $\nabla \cdot g$ is Γ -equivariant, and gives rise to the same point in the moduli space of flat (possibly singular) $G_{\mathbb{C}}$ -connections on Σ/Γ .*

The above lemma follows from the definition of Γ -equivariant flat connections, and therefore by Propositions 2, 3 and 5, the connected components Γ -equivariant flat irreducible $G_{\mathbb{C}}$ -connections can be locally identified with open subsets of the moduli space of flat irreducible connections on Σ/Γ with monodromies in fixed conjugacy classes determined by the branch order of $\Sigma \rightarrow \Sigma/\Gamma$ and by the component. One should note that the corresponding moduli spaces of irreducible flat connections on Σ and Σ/Γ are in general not globally the same, as can be seen in the following example.

Example 6.1. Consider a Riemann surface Σ of genus 3 which admits a fix point free involution $\psi: \Sigma \rightarrow \Sigma$ giving a double covering to a Riemann surface $M := \Sigma/\mathbb{Z}_2$ of genus 2. Let ∇ be a flat unitary line bundle connection such that ∇ is not self-dual, i.e., ∇^* is not gauge equivalent to ∇ , and such that $\psi^* \nabla = \nabla^*$. Then, $\nabla \oplus \nabla^*$ is a \mathbb{Z}_2 -equivariant flat connection, whose corresponding flat connection on Σ/\mathbb{Z}_2 is irreducible. A more extensive analysis of this set up is given in Section 6 to illustrate the results of the paper.

As mentioned previously, by Hitchin's work [14] and generalizations thereof, the space of irreducible flat $G_{\mathbb{C}}$ -connections is a hyper-Kähler manifold. In this context, one has the following results.

Theorem 7. *Let Σ be a compact Riemann surface of genus $g \geq 2$ and Γ be a finite group acting on Σ by holomorphic automorphisms. Then, the connected components of the space of gauge equivalence classes of irreducible Γ -equivariant flat $G_{\mathbb{C}}$ -connections are hyper-Kähler submanifolds of the moduli space of flat irreducible $G_{\mathbb{C}}$ -connections on Σ .*

Proof. By Lemma 6 any connected component of the space of gauge equivalence classes of irreducible Γ -equivariant flat $G_{\mathbb{C}}$ -connections is a complex submanifold with respect to the complex structure J coming from the complex group $G_{\mathbb{C}}$. In order to show that it is also a complex submanifold with respect to the complex structure I (coming from solutions of the self-duality equations on the Riemann surface), one needs to make use of the uniqueness of solutions. Fixing a trivialization of the underlying C^{∞} bundle in order to work on $\mathbb{C}^n \rightarrow \Sigma$, consider the standard hermitian metric h on $\mathbb{C}^n \rightarrow \Sigma$ which is invariant under Γ . Consider ∇ an irreducible Γ -equivariant flat $G_{\mathbb{C}}$ -connection such that (∇, h) solves the self-duality equations, i.e., so that $\nabla = \nabla^u + \Phi + \Phi^*$ where ∇^u is unitary with respect to h and Φ^* is the adjoint of the holomorphic Higgs field Φ (with respect to $(\nabla^u)^{(0,1)}$). Now, for $\psi \in \Gamma$ one has that $\psi^*\nabla = \nabla \cdot g_{\psi}$ decomposes into harmonic parts as $\psi^*\nabla = \psi^*\nabla^u + \psi^*\Phi + \psi^*\Phi^*$, since h is Γ -invariant. Then, from the uniqueness of solutions of the self-duality equations (e.g. see [14] in the case of $G_{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$), the gauge transformation g_{ψ} must already be unitary. Therefore, since ∇ is Γ -equivariant, the corresponding Higgs pair $((\nabla^u)^{(0,1)}, \Phi)$ is also Γ -equivariant (in the same sense as defined for connections), which proves the theorem. \square

Remark 8. From [24, Corollary 3.9], given any orbifold Riemann surface $\tilde{\Sigma}$ with negative Euler characteristic, there exists a smooth compact Riemann surface Σ with an action of a finite group Γ , such that $\tilde{\Sigma} = \Sigma/\Gamma$, and thus the analysis done in this paper could be translated in terms of Higgs bundles on orbifold Riemann surfaces.

From Remark 8, properties of the (B, B, B) -branes constructed through finite group actions can be deduced from [24]. From Theorem 7 and [24, Section 3D] one can describe the spaces of Γ -equivariant flat $G_{\mathbb{C}}$ -connections as (B, B, B) -branes also in the case of actions with no fixed points. Indeed, consider $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C}), \mathrm{GL}(n, \mathbb{C})$: in these cases the spaces of gauge equivalence classes of irreducible Γ -equivariant flat $G_{\mathbb{C}}$ -connections are hyper-Kähler submanifolds of the moduli spaces of flat irreducible $G_{\mathbb{C}}$ -connections on Σ , and are naturally covered by an open dense subset of the hyper-Kähler moduli space of irreducible flat $G_{\mathbb{C}}$ -connections (or possibly $\mathrm{PSL}(n, \mathbb{C})$ -connections, as in Remark 4) on the surface Σ/Γ .

Remark 9. Through the monodromy of rank 2 Higgs bundles given in [3] on equivariant points, one can deduce connectivity of the equivariant branes described in this section.

2.3. Equivariant (A, B, A) -branes. By considering a real structure $f : \Sigma \rightarrow \Sigma$ on the Riemann surface, it was shown in [4, 5] how to construct and study families of (A, B, A) -branes in the moduli space of $G_{\mathbb{C}}$ -Higgs bundles. In particular, for ξ the compact anti-holomorphic involution of $G_{\mathbb{C}}$, the fixed point set of $i_2(\bar{\partial}_A, \Phi) := (f^*(\partial_A), f^*(\Phi^*)) = (f^*(\xi(\bar{\partial}_A)), -f^*(\xi(\Phi)))$, defines an (A, B, A) -brane which lies in the Hitchin fibration for $G_{\mathbb{C}}$ -Higgs bundles $(\bar{\partial}_A, \Phi)$ as a real integrable system [4, Theorem 17]. Moreover, for Σ a hyperelliptic curve of genus 3, from [10, Section 6] and [4, Appendix A] one can deduce further characteristics of the brane. Therefore, these Riemann surfaces shall be taken as toy models along this paper.

Following Section 2.2, one may consider both an orientation reversing involution with fixed points, as well as a finite group action on the Riemann surface Σ . By taking both involutions together, and looking at the induced action on the moduli space of flat connections, one would obtain the intersection of a (B, B, B) -brane and an (A, B, A) -brane. The compatibility and classification of these involutions is done in [7], the (B, B, B) -branes would be as described in Section 2.2, and the (A, B, A) -branes as described in [4]. Therefore, by considering these two actions, one obtains natural mid-dimensional (A, B, A) -branes inside the (B, B, B) -brane, which shall be referred to as CMC-branes. Finally, in [8, Section 8] the equivariant cohomology for equivariant bundles is calculated in terms of the action of G on the usual cohomology, providing the tools to understand the cohomology of the CMC-branes.

3. MID-DIMENSIONAL EQUIVARIANT BRANES

Mid-dimensional subspaces of the moduli space $\mathcal{M}_{G_{\mathbb{C}}}$ of $G_{\mathbb{C}}$ -Higgs bundles appear to be of particular interest, as these can be both B -branes and A -branes (see, for example, the families constructed in [4, 5]). Thus, in what follows mid-dimensional hyper-Kähler subspaces coming from Γ -equivariant flat $G_{\mathbb{C}}$ -connections shall be described, and a study of different finite group actions on Σ which lead to mid-dimensional equivariant branes shall be carried through.

Whilst results here can be seen for higher rank groups, the present paper shall focus on the group $G_{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$. We shall denote by \mathcal{M}_g the moduli space of flat $\mathrm{SL}(2, \mathbb{C})$ -connections on a compact Riemann surface of genus g , and for simplicity drop the group label, but maintain the label for the genus since it will become of use at several stages of our analysis. Recall that the dimension of this space is $\dim \mathcal{M}_g = 6g - 6$, and the dimension of the moduli space $\mathcal{M}_{\gamma, n}$ of flat $\mathrm{SL}(2, \mathbb{C})$ -connections on a n -punctured Riemann surface of genus γ with fixed local monodromies (with simple eigenvalues) is

$$(5) \quad \dim \mathcal{M}_{\gamma, n} = 6\gamma - 6 + 2n,$$

provided that either $\gamma \geq 2$, or that $\gamma \geq 1$ and $n \geq 1$, or finally that $\gamma = 0$ and $n \geq 4$. Otherwise, the dimension of $\mathcal{M}_{\gamma, n}$ is either 0 or 2. Note that the corresponding moduli spaces of flat $\mathrm{PSL}(n, \mathbb{C})$ -connections have the same dimensions as their $\mathrm{SL}(n, \mathbb{C})$ counterparts. In order to study mid-dimensional subspaces of \mathcal{M}_g coming from finite group actions, note that the dimension of the space of Γ -equivariant $\mathrm{SL}(2, \mathbb{C})$ -connections and the order of the group Γ are closely related.

Proposition 10. *Let Σ be a compact Riemann surface of genus $g \geq 2$, and Γ be a finite group of order h acting on Σ by holomorphic automorphisms. If a component of the moduli space of equivariant flat $\mathrm{SL}(2, \mathbb{C})$ -connections on Σ has half the dimension of the moduli space \mathcal{M}_g , then $h = 2^k$ for some $k \in \mathbb{N}$.*

Proof. In order to show that all prime factors of h are 2, note that for each prime factor p of h there exists a subgroup Γ_p of Γ such that $\Gamma_p \cong \mathbb{Z}_p$. Considering the induced group action $\mathbb{Z}_p \times \Sigma \rightarrow \Sigma$, by Riemann-Hurwitz the genus g and γ of the Riemann surfaces Σ and Σ/\mathbb{Z}_p , respectively, satisfy

$$(6) \quad n(p-1) = 2(g-1 + p(1-\gamma)).$$

Thus since $\mathcal{M}_g = 6g - 6$ one has that

$$(7) \quad \dim \mathcal{M}_g = 3n(p-1) + 6p(\gamma-1).$$

From Section 2, any component of the moduli space of flat Γ -equivariant $\mathrm{SL}(2, \mathbb{C})$ -connections on Σ can be identified with the moduli space of flat $\mathrm{SL}(2, \mathbb{C})$ or $\mathrm{PSL}(2, \mathbb{C})$ -connections on a n -punctured compact Riemann surface of genus γ , for $n \in \mathbb{N}$ satisfying (6), and hence

$$(8) \quad 3n(p-1) + 6p(\gamma-1) \leq 12(\gamma-1) + 4n.$$

Moreover, the inequality is also valid in the special case of invariant flat connections, where one has to consider \mathcal{M}_{γ} instead of $\mathcal{M}_{\gamma, n}$. One should note that the exceptional cases in (5), i.e. the case of $\gamma = 0$ and $n \geq 3$, and of $\gamma = 1$ and $n = 0$, are excluded by dimensional reasons: indeed, in these cases $\dim \mathcal{M}_g \geq 6 > 4 \geq 2\dim \mathcal{M}_{\gamma, n}$.

When the genus is $\gamma > 1$, the inequality (8) only holds if $p = 2$. Similarly, when the genus is $\gamma = 1$, equation (8) holds only when $p = 2$ (since $\gamma = 1$ and $n = 0$ imply that $g = 1$ which would contradict the assumption that $g \geq 2$). Finally, when $\gamma = 0$ the inequality (8) is equivalent to

$$(9) \quad (3p-7)n \leq 6(p-2),$$

which holds only when $p = 2$, since when $n = 2$ the Riemann surface Σ would have genus 0 which would contradict again the assumption that $g \geq 2$. \square

Theorem 11. *Let Σ be a compact Riemann surface of genus $g \geq 2$ and Γ be a finite group acting on Σ by holomorphic automorphisms such that a component of the moduli space of Γ -equivariant flat $\mathrm{SL}(2, \mathbb{C})$ -connections on Σ has half the dimension of the moduli space \mathcal{M}_g . Then one of the following holds:*

- (I) $\Gamma = \mathbb{Z}_2$ acts by a fix-point free involution τ on Σ , or
- (II) Σ is hyperelliptic of genus 3 and $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$.

In the later case (II), one of the \mathbb{Z}_2 -factors corresponds to the hyperelliptic involution ψ , whilst the other \mathbb{Z}_2 -factor corresponds to an involution ρ with 4 fixed points.

Proof. By Proposition 10, the order of Γ must be 2^k for some $k \in \mathbb{N}$. When $k = 1$, equality in (8) implies that the number n of fix points is 0 and one recovers the case (I). Consider then $k \geq 2$, in which case the finite group Γ contains a subgroup of order 4 isomorphic to \mathbb{Z}_4 or $\mathbb{Z}_2 \times \mathbb{Z}_2$. But if \mathbb{Z}_4 is a subgroup of Γ , the same arguments as in the proof of Proposition 10 apply for \mathbb{Z}_p with $p = 4$, leading to a contradiction. Hence, the group Γ must contain $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a subgroup.

In what follows the two generators of the \mathbb{Z}_2 -actions on Σ shall be denoted by ψ and ρ . Since the stabilizer subgroup of a point in the Riemann surface Σ is cyclic, the fixed points of ψ and ρ must be distinct. Consider then

$$\Sigma_\psi := \Sigma/\psi = \Sigma/\mathbb{Z}_2; \quad \text{with } g_\psi := \text{genus of } \Sigma_\psi,$$

and n_ψ the number of fixed points of ψ . Then, by Riemann-Hurwitz, $n_\psi = 2g + 2 - 4g_\psi$, and hence the genus $g_{\rho, \psi}$ of the quotient $\Sigma_\psi/\mathbb{Z}_2 = \Sigma/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is given by

$$(10) \quad g_{\rho, \psi} = \frac{1}{8}(6 + 2g - n_\rho - n_\psi),$$

where n_ρ is the number of fixed points of ρ (acting on Σ). Thus, as in Proposition 10, since $g_{\rho, \psi} \geq 0$ and $n_\rho, n_\psi \geq 0$, one must have

$$(11) \quad 6 + 2g \geq n_\rho + n_\psi \geq 6g - 6.$$

From equations (10)-(11) one therefore has that $2 \leq g \leq 3$. Note that by Theorem 7 the real dimension of the moduli space needs to be divisible by 8, and thus the genus g must be $g = 3$. In this case, equations (10)-(11) imply that $n_\rho + n_\psi = 12$. Moreover, as involutions on a surface of genus 3 can not have 12 fixed points by Riemann-Hurwitz, it follows that either the involutions have $n_\psi = 8$ and $n_\rho = 4$ fixed points, or $n_\psi = 4$ and $n_\rho = 8$. Therefore, Σ must be hyperelliptic and, without loss of generality, one may consider ψ to be the hyperelliptic involution and ρ an involution with 4 fixed points. Finally, $\mathbb{Z}_2 \times \mathbb{Z}_2$ must be equal to Γ since otherwise the dimension of the moduli space of equivariant $\mathrm{SL}(2, \mathbb{C})$ -connections would be strictly less than 6, which is a contradiction. \square

Remark 12. *The connections on the quotient space of Theorem 11 (II) must have local monodromies conjugated to $\mathrm{diag}(1, -1)$ around any of the 6 branch values in \mathbb{CP}^1 , and thus are $\mathrm{PSL}(2, \mathbb{C})$ -connections, since otherwise the dimension of the corresponding moduli space would be strictly less than 6.*

One should note that a genus 3 curve Σ is either hyperelliptic or it is a non-singular plane quartic. In the hyperelliptic case Σ has exactly 8 Weierstrass points which are the ramification points of the canonical map $\Sigma \rightarrow \mathbb{CP}^1$, and the action of $\mathrm{Aut}(\Sigma)$ on the Weierstrass can be found in [20, Table 3]. More information about these actions can be found in [2] and references therein. Moreover, $\#\mathrm{Aut}(\Sigma) \leq 168$, and thus only 2, 3 and 7 can divide the order of $\mathrm{Aut}(\Sigma)$.

In the case of Theorem 11 (II) the quotient $\Sigma/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is the sphere with 6 marked points: A pair of points corresponds to the 4 fixed points of ρ while the remaining 4 marked points are the image of the 8 Weierstrass points of Σ . The space of surfaces Σ as in Theorem 11 (II) is complex

3-dimensional, and local coordinates are given by 3 pairwise distinct points on $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$. The space of holomorphic quadratic differentials on Σ which are invariant under $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ is (naturally isomorphic to) the space of meromorphic quadratic differentials on \mathbb{CP}^1 with at most simple poles at the 6 marked points, which is also complex 3-dimensional. One should note that Example 6.1 fits in this case, and further study of this setting in terms of Higgs bundles shall be given in Section 6.

4. HIGGS BUNDLES AND THE HITCHIN FIBRATION

In order to understand the geometry and topology of the mid-dimensional branes constructed in Section 2.2, these branes shall be studied through Higgs bundles. Recall that Higgs bundles appeared in N. Hitchin's work as solutions of Yang-Mills self-duality equations on a Riemann surface [14]. Classically, a *Higgs bundle* on a compact Riemann surface Σ of genus $g \geq 2$ is a pair (E, Φ) where E is a holomorphic vector bundle on Σ , and Φ , the *Higgs field*, is a holomorphic 1-form in $H^0(\Sigma, \text{End}_0(E) \otimes K_\Sigma)$, for K_Σ the cotangent bundle of Σ and $\text{End}_0(E)$ the traceless endomorphisms of E . Higgs bundles can also be defined for complex groups G_c , and through stability conditions, one can construct their moduli spaces \mathcal{M}_{G_c} .

A natural way of studying the moduli space of Higgs bundles is through the *Hitchin fibration*, sending the class of a Higgs bundle (E, Φ) to the coefficients of the characteristic polynomial $\det(xI - \Phi)$. The generic fibre is an abelian variety, which can be seen through line bundles on an algebraic curve S , the *spectral curve* associated to the Higgs field [15]. For instance in the case of classical $\text{GL}(n, \mathbb{C})$ -Higgs bundles, the Hitchin base is $\bigoplus_{i=1}^n H^0(\Sigma, K_\Sigma^i)$ and the smooth fibres can be seen through *spectral data* as Jacobian varieties of S , and in the case of $\text{SL}(n, \mathbb{C})$ -Higgs bundles the Hitchin base is $\bigoplus_{i=2}^n H^0(\Sigma, K_\Sigma^i)$ and the generic fibres are given by $\text{Prym}(S, \Sigma)$.

When considering Higgs bundles fixed by involutions, one can study the induced action on the Hitchin fibration to understand those Higgs bundles (see, for instance [16] for Higgs bundles for split real forms, and more generally [25] for any real form, and [4, 5] for other involutions). In what follows the induced action of the finite group Γ on Higgs bundles shall be considered first, in order to later describe how the (B, B, B) -branes from Theorem 7 intersect the Hitchin fibration. Then, through the duality of abelian varieties in the fibres of the Hitchin fibration, we shall comment on the dual (B, A, A) -branes in Section 7. Since the (B, B, B) -branes from Theorem 7 appearing as mid-dimensional spaces in the moduli space of $\text{SL}(2, \mathbb{C})$ -Higgs bundles correspond to Higgs bundles whose spectral curves are double covers of Σ , some basic facts about unbranched and branched double covers of a Riemann surface Σ shall be mentioned next.

Consider a flat vector bundle $E \rightarrow \Sigma$, endowed with a smooth action of a linear group G , such that the projection is G -equivariant. The group G acts naturally on the space of differential forms $\Omega^*(\Sigma, E)$ on Σ with values in E . The vector bundle E is called *G -equivariant flat vector bundle* if $g \circ \nabla = \nabla \circ g$ and $\nabla_{X_\Sigma} = \mathcal{L}^E(X)$ for any $g \in G$ and any $X \in \mathfrak{g}$, where ∇ is the covariant derivative of E , and $\mathcal{L}^E(X) : \Omega^*(\Sigma, E) \rightarrow \Omega^*(\Sigma, E)$ is the corresponding Lie algebra action and X_Σ is the vector field on Σ defined by the action of \mathfrak{g} . If the group G is connected, then $g \circ \nabla = \nabla \circ g$ follows from $\nabla_{X_\Sigma} = \mathcal{L}^E(X)$. If G is finite, the latter two equalities follow from each other. However, these conditions are independent in general.

Unbranched double covers are known to be parametrized by $H_1(\Sigma, \mathbb{Z}_2)$ (e.g. [21]), torsion two line bundles P_2 on Σ . Then, for any $\alpha \in H_1(\Sigma, \mathbb{Z}_2)$ there exists a unique flat connection ∇ (up to gauge transformation) such that its monodromy $m_\nabla : \pi_1(\Sigma) \rightarrow H_1(\Sigma, \mathbb{Z}) \rightarrow \mathbb{C}^*$, which is abelian, is given by α . The parallel transport along (not necessarily closed) curves γ from $q \in \Sigma$ to $q' \in \Sigma$ shall be denoted by $f_\gamma : (P_2)_q \rightarrow (P_2)_{q'}$. Fixing a point s_0 in the fibre $(P_2)_{q_0} \setminus \{0\}$, for some $q_0 \in \Sigma$, the double cover corresponding to ∇ is

$$(12) \quad S^\alpha := \{s_q \in (P_2)_q \mid q \in \Sigma; \exists \gamma \text{ from } q_0 \text{ to } q \text{ with } f_\gamma(s_0) = s_q\}$$

where the covering map $S^\alpha \rightarrow \Sigma$ is defined by $s_q \mapsto q$. Note that in particular α is trivial if and only if S^α is not connected.

Branched double covers can be constructed through holomorphic sections $s \in H^0(\Sigma, L^2)$, for L a holomorphic line bundle on Σ . For simplicity, one may restrict to sections which have simple zeros¹. Then, there is a (unique) double cover $\pi: S \rightarrow \Sigma$ branched over the zeros of s such that a square root $t \in H^0(S, \pi^*L)$ satisfying $t^2 = s$ exists. The cover is then given by

$$(13) \quad S = \{t_q \in L_q \mid t_q^2 = s_q\}.$$

Two double covers $\pi: S \rightarrow \Sigma$ and $\tilde{\pi}: \tilde{S} \rightarrow \Sigma$ are said to differ by the flat (holomorphic) \mathbb{Z}_2 -bundle P_2 if they correspond to the same holomorphic section $s \in H^0(\Sigma, L^2)$ but to the line bundles L and $P_2 \otimes L$ respectively.

5. EQUIVARIANT BRANES AND FIXED POINT FREE ACTIONS

As seen in Theorem 11 (I), mid dimensional equivariant (B, B, B) branes in the moduli space of Higgs bundles can be constructed through fixed point free actions on the Riemann surface Σ . Thus, in what follows, these induced branes shall be seen inside the Hitchin fibration.

5.1. Fix point free actions and equivariant Higgs bundles. Let Σ be a Riemann surface of genus $g = 2\gamma - 1$ with a fix point free involution τ , and $\Sigma_\tau := \Sigma/\tau$ its quotient, which has genus γ . Every τ -invariant holomorphic quadratic differential Q on Σ is the pull-back of a holomorphic quadratic differential Q_τ on Σ_τ , and as done previously, we shall restrict to those with only simple zeros (i.e., to generic points in the Hitchin base). Consider S and S_τ the double covers of Σ and Σ_τ defined by Q and Q_τ , respectively. The involution τ lifts to a fix point free involution on S , denoted by the same symbol, and $S/\tau = S_\tau$. Moreover, the involution τ and the involution σ switching the sheets of the double cover S commute.

Theorem 13. *Let τ be a fix point free involution as in Theorem 11. The τ -equivariant (B, B, B) -brane intersects a generic fibre of the Hitchin fibration over a point defining the spectral curve S in the abelian variety $\mathcal{P} := \{L \in \text{Jac}(S) \mid \sigma^*L = L^*, \tau^*L = L\}$ given by*

$$(14) \quad \mathcal{P} = \text{Prym}(S_\tau, \Sigma_\tau)/\mathbb{Z}_2 \subset \text{Prym}(S, \Sigma),$$

where the generator of \mathbb{Z}_2 is the holomorphic line bundle P_2 on S_τ corresponding to the (unbranched) covering $S \rightarrow S_\tau$.

Proof. From [14], the $SL(2, \mathbb{C})$ Hitchin base is given by $H^0(\Sigma, K_\Sigma^2)$, and a generic fibre of the Hitchin fibration over $Q \in H^0(\Sigma, K_\Sigma^2)$ with simple zeros is given by the Prym variety

$$(15) \quad \{L \in \text{Jac}(S) \mid \sigma^*L \otimes L = \mathcal{O}\}.$$

If a Higgs pair (E, Φ) is equivariant with respect to the automorphism τ , then the corresponding eigenline bundle L must be symmetric, i.e., $\tau^*L = L$. In order to show that the space is $\mathcal{P} = \text{Prym}(S_\tau, \Sigma_\tau)/\mathbb{Z}_2$ note that via pull-back one has a surjective map

$$(16) \quad \text{Prym}(S_\tau, \Sigma_\tau) \rightarrow \mathcal{P} \subset \text{Prym}(S, \Sigma).$$

Surjectivity can be seen from looking at the dimensions of the corresponding abelian varieties, which are the same by construction. It remains to compute the kernel of the map (16). Let L be a holomorphic line bundle of degree 0 on S_τ which pulls back to the trivial holomorphic line bundle on S , and equip the line bundle with its unique compatible unitary flat connection ∇ . On S , this flat connection is trivial. Thus ∇ has monodromy ± 1 along closed curves on S_τ , and -1 is only possible when a lift of a loop to S does not close. Therefore, the line bundle L is either the trivial bundle, or the bundle P_2 corresponding to the covering $S \rightarrow S_\tau$. \square

¹This condition will become apparent and natural when considering the Hitchin fibration, since the smooth loci is given by sections in $H^0(\Sigma, K^2)$ with simple zeros.

5.2. Fix point free actions and anti-equivariant Higgs bundles. Let Σ be a Riemann surface of odd genus $g = 2\gamma - 1$ with a fix point free involution τ as in Theorem 11. Following Section 3, *anti-equivariant Higgs bundles* $(\bar{\partial}, \Phi)$ with respect to the involution τ are given by a τ -equivariant holomorphic structure $\bar{\partial}$ and a Higgs field Φ which satisfies

$$\tau^* \Phi = -g^{-1} \circ \Phi \circ g,$$

where g is the gauge transformation such that $\tau^* \bar{\partial} = \bar{\partial} \circ g$ and $(g \circ \tau)g = \text{id}$. In this situation, the following analog of Theorem 7 can be proved:

Proposition 14. *The connected components of the moduli space of stable anti-equivariant Higgs bundles are complex submanifolds of the moduli space of Higgs bundles with respect to I and Lagrangian with respect to J and K .*

Proof. Over the locus of stable bundles, the full moduli space is the cotangent bundle and the complex structure decouples. The moduli space of equivariant stable bundles is a complex submanifold of the moduli space of stable bundles, and anti-equivariant Higgs fields for a τ -equivariant bundle are a complex subvector space of the space of Higgs fields. Hence, gauge classes of anti-equivariant Higgs bundles give rise to a complex submanifold. To see that the holomorphic symplectic form vanishes on this submanifold, we just apply the transformation formula for the diffeomorphism τ to the integral defining the holomorphic symplectic structure. The dimension is easily shown to be half of the dimension of the full moduli space. \square

Note that for Φ a Higgs field with spectral line bundle L , the Higgs field $-\Phi$ has spectral line bundle $\sigma^* L$. Thus, following similar steps as in the proof of Theorem 13, one can see that an anti-equivariant Higgs pair is determined (after choosing a square root of its determinant and after fixing a τ -invariant base point P_2 in the Prym variety) by a point in the space

$$(17) \quad \mathcal{P}^\vee := \{L \in \text{Jac}(S) \mid \sigma^* L = L^*, \tau^* L = L^*\} \subset \text{Prym}(S, \Sigma).$$

Remark 15. *When considering involutions on the spectral data associated to $SL(2, \mathbb{C})$ -Higgs bundles, from [25, Theorem 4.12] one has that line bundles in $\text{Prym}(S, \Sigma)$ of order two induce certain real Higgs bundles, namely $SL(2, \mathbb{R})$ -Higgs bundles. These are the line bundles which are both invariant and anti-invariant with respect to the involution defining the cover $S \rightarrow \Sigma$, and the monodromy of the fibration has an explicit description in terms of spectral data [26].*

As seen before, the natural involution σ of the spectral curve S and the fixed point free involution τ commute on S . Hence, $\tau \circ \sigma$ is an involution which is fix point free and it descends to an involution in the quotient Riemann surface $\tilde{S} := S/(\tau \circ \sigma)$. Moreover, the quotient Riemann surface \tilde{S}/σ becomes Σ_τ in a natural way yielding the following diagram:

$$(18) \quad \begin{array}{ccccc} & & S & & \\ & \swarrow \text{mod } \sigma & \downarrow & \searrow \text{mod } \tau \circ \sigma & \\ \Sigma & & S_\tau := S/\tau & & \tilde{S} \\ & \swarrow \text{mod } \tau & \downarrow & \searrow \text{mod } \sigma & \\ & & \Sigma_\tau = (S_\tau)/\sigma & & \end{array}$$

Proposition 16. *The abelian variety \mathcal{P}^\vee in (17) is given by $\mathcal{P}^\vee = \text{Prym}(\tilde{S}, \Sigma_\tau)/\mathbb{Z}_2$ where the generator of \mathbb{Z}_2 is the holomorphic torsion-2 line bundle \tilde{P}_2 on \tilde{S} defining the cover $S \rightarrow \tilde{S}$.*

Proof. Let \tilde{P} be a fixed base point in the affine Prym which is in \mathcal{P} . Since σ and τ commute on S , one has that $\mathcal{P}^\vee = \{L \in \text{Jac}(S) \mid \sigma^* L = L^*, (\sigma \circ \tau)^* L = L\} \subset \text{Prym}(S, \Sigma)$, so the proposition follows analogously to the proof of Theorem 13. \square

6. HYPERELLIPTIC SURFACES OF GENUS 3

As seen in Theorem 11, equivariant branes on Riemann surfaces Σ of genus 3 occur either with a fix point free involution τ (as studied in Section 5), or with a hyperelliptic involution ψ and a second involution ρ which has 4 fix points. In what follows both settings for compact Riemann surfaces of genus 3 shall be considered.

6.1. Equivariant branes through ψ and ρ . In order to study the latter case appearing in Theorem 11 (II), consider a compact Riemann surface Σ of genus 3 and a holomorphic quadratic differential Q with simple zeros which is invariant under the involutions ψ and ρ giving the a finite group action of Γ as in Theorem 11 (II). In what follows, we shall study the intersection \mathcal{P} of the mid-dimensional (B, B, B) brane of equivariant Higgs bundles in Theorem 11 with the generic fibres of the $SL(2, \mathbb{C})$ Hitchin fibration over $Q \in H^0(\Sigma, K_\Sigma)$ with simple zeros. The double cover S defined in (13) giving the spectral curve of Higgs fields Φ for which $\det(\Phi) = Q$ has genus 9 and is defined as

$$(19) \quad \pi: S = \{\omega_p \in K_p \mid p \in \Sigma; -\omega_p^2 = Q_p\} \rightarrow \Sigma.$$

The cover inherits the natural involution $\sigma: S \rightarrow S$ with $\pi \circ \sigma = \pi$, and the involutions ψ and ρ lift to commuting involutions (denoted by the same symbols) on S . Note that neither of the involutions ψ and ρ on S have fixed points since the points over the fixed points on Σ are interchanged. As in the case of fixed point free involutions, one may consider the quotient

$$(20) \quad \tilde{S} = S/(\mathbb{Z}_2 \times \mathbb{Z}_2),$$

which is now a hyperelliptic surface of genus 3, not necessarily the same as Σ . Its hyperelliptic involution is given by the induced action of σ , and it branches over the 8 points on \mathbb{CP}^1 : six of them are the marked points of the sphere as in Section 3, while the other two branch points are given by the two zeros of the meromorphic quadratic differential on \mathbb{CP}^1 which pulls back to Q .

As mentioned previously, the intersection $\mathcal{P} \subset \text{Prym}(S, \Sigma)$ of the mid-dimensional (B, B, B) brane of equivariant Higgs bundles with a generic fibre $\text{Prym}(S, \Sigma)$ of the $SL(2, \mathbb{C})$ Hitchin fibration parametrizes the $(\psi$ - and ρ -) equivariant Higgs fields with determinant Q . In order to give a geometric description of this variety, note the following:

Proposition 17. *The Jacobian $\text{Jac}(\tilde{S})$ is immersed (via pull-back) into $\text{Prym}(S, \Sigma)$, and the corresponding kernel is finite.*

Proof. The last part of the proposition is clear once one has the existence of a line bundle $M \rightarrow S$ satisfying $M^2 = K_\Sigma^*$ and which is invariant under ψ and ρ . Such a bundle needs to be the pull-back of $\mathcal{O}(-1) \rightarrow \mathbb{CP}^1$ by the fourfold covering $S \rightarrow \mathbb{CP}^1$ given by factoring out σ and ψ . Given $L \in \text{Jac}(\tilde{S})$, one has that $L \otimes \sigma^* L$ is invariant under σ , and hence it is the pull-back of a holomorphic line bundle on $\mathbb{CP}^1 = \tilde{S}/\sigma$. Moreover, since the cover is obtained through the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$, the monodromy of the cover

$$(21) \quad p: H_1(\tilde{S}, \mathbb{Z}) \rightarrow \mathcal{S}_4$$

is abelian, for \mathcal{S}_4 the 4-symmetric group. Furthermore, the action is of order two: for each $\gamma \in H_1(\tilde{S}, \mathbb{Z})$ the composition is $p(\gamma) \circ p(\gamma) = \text{Id}$. In order to see when the pull-back of a holomorphic line bundle $L \in \text{Jac}(\tilde{S})$ is trivial on S , equip L with its unique compatible unitary flat connection ∇^L . The bundle L is in the kernel of the pull-back map if and only if the monodromy representation of the pull-back of ∇^L on S is trivial. This happens if and only if the monodromy of ∇^L along a closed curve γ is 1 when $p(\gamma) = \text{Id}$, and ± 1 if $p(\gamma) \neq \text{Id}$, and the set of such line bundles is finite. \square

Following the notation from previous sections, consider $\Sigma_\rho := \Sigma/\rho$ and $S_\rho := S/\rho$. Then, one has the following:

Proposition 18. *The branched covers $S/(\rho \circ \sigma) \rightarrow \Sigma_\rho$ and $S_\rho \rightarrow \Sigma_\rho$ differ by the \mathbb{Z}_2 -bundle determining the unbranched cover $\Sigma \rightarrow \Sigma_\rho$.*

Proof. In order to prove the proposition, consider the concrete description of 2-fold covers in Section 4. The surface Σ is given by pairs (q, s_q) where $q \in \Sigma$ and s_q is given by parallel transport along some curve (with fixed start point) and end point q with respect to the unitary flat connection corresponding to the \mathbb{Z}_2 -bundle $P_2 \rightarrow \Sigma_\rho$. From that perspective the spectral curve S is given by triples (q, s_q, ω_q) where (q, s_q) are as above and $\omega_q \in K_{\Sigma_\rho}$ satisfies $\omega_q^2 = (Q_\rho)_q$. This identification holds as the pull-back of K_{Σ_ρ} to Σ is the canonical bundle K_Σ . Then, the involutions σ and ρ act as

$$(22) \quad \sigma : (q, s_q, \omega_q) \mapsto (q, s_q, -\omega_q),$$

$$(23) \quad \rho : (q, s_q, \omega_q) \mapsto (q, -s_q, \omega_q).$$

The spectral curve S_ρ is thus obtained by identifying $(q, s_q, \omega_q) \sim (q, -s_q, \omega_q)$, and the curve $S/(\rho \circ \sigma)$ by identifying $(q, s_q, \omega_q) \sim (q, -s_q, -\omega_q)$. By taking the tensor product, i.e., $(q, s_q \otimes \omega_q)$, which is well-defined on $S/(\rho \circ \sigma)$, it follows that the branched cover $S/(\rho \circ \sigma) \rightarrow \Sigma_\rho$ is determined by $Q_\rho \in H^0(\Sigma_\rho, K_{\Sigma_\rho}^2)$ and the holomorphic square root $P_2 \otimes K_{\Sigma_\rho}$ of $K_{\Sigma_\rho}^2$ as required. \square

The setting of Theorem 11 (II) can be shown to be equivalent to the one of Theorem 11 (I) for Riemann surfaces of genus 3, reducing the study of the mid-dimensional equivariant (B, B, B) branes to fixed point free actions on Riemann surfaces:

Lemma 19. *Let Σ be a hyperelliptic Riemann surface of genus 3 with hyperelliptic involution by ψ , equipped with an additional involution ρ with 4 fix points. Then, $\tau = \rho \circ \psi$ is a fix point free involution.*

Proof. Note that since ρ and ψ commute, the map ρ is an involution. Moreover, ρ gives rise to an involution on the quotient $\Sigma/\psi = \mathbb{CP}^1$, which must have exactly two fix points by Riemann-Hurwitz. The possible fix points of ρ on Σ must map to the fix points of ρ on \mathbb{CP}^1 . Hence, there are only 4 possible fix points, but these are already fix points of ρ by assumption, and therefore they are interchanged by $\tau = \psi \circ \rho$. \square

6.2. Equivariant branes through a fix point free involution τ . From the previous analysis, the hyperelliptic involution ψ on a compact Riemann surface Σ of genus $g = 3$ together with a fix point free involution τ also induce an involution $\rho = \psi \circ \tau$ which has 4 fixed points. Moreover, one can see that all genus 3 Riemann surfaces with fixed point free actions must be hyperelliptic:

Proposition 20. *Let Σ be a Riemann surface of genus 3 with a fix point free involution τ . Then, Σ is hyperelliptic.*

The proof is analogous to the proof of Lemma 19, and thus a hyperelliptic Riemann surface of genus 3 with an additional involution with 4 fix points is the same as Riemann surface of genus 3 with a fix point free involution. Hence, the equivariant points in the Hitchin base can be described as follows:

Proposition 21. *Let Σ be a hyperelliptic Riemann surface of genus 3 with an additional involution ρ with 4 fix points. Then, a holomorphic quadratic differential is invariant under the hyperelliptic involution ψ and invariant under ρ if and only if it is invariant under $\tau = \rho \circ \psi$.*

Proof. Without loss of generality, suppose that Σ is given by the algebraic equation

$$y^2 = (z - z_1)(z + z_1)(z - z_2)(z + z_2)(z - z_3)(z + z_3)(z - z_4)(z + z_4),$$

for 8 pairwise disjoint points $\pm z_1, \dots, \pm z_4 \in \mathbb{C} \setminus \{0\} \subset \mathbb{CP}^1$, and that $\psi : (y, z) \mapsto (y, -z)$, or, equivalently, given by $\rho(y, z) = (-y, -z)$. A basis for the 6-dimensional space $H^0(\Sigma, K_\Sigma^2)$ is then

$\left\{ \frac{1}{y^2}(dz)^2, \frac{z}{y^2}(dz)^2, \frac{z^2}{y^2}(dz)^2, \frac{z^3}{y^2}(dz)^2, \frac{z^4}{y^2}(dz)^2, \frac{1}{y}(dz)^2 \right\}$. Therefore the space of τ -invariant holomorphic quadratic differentials is spanned by $\left\{ \frac{1}{y^2}(dz)^2, \frac{z^2}{y^2}(dz)^2, \frac{z^4}{y^2}(dz)^2 \right\}$, and this is exactly the space of ψ and ρ invariant holomorphic quadratic differentials. \square

From the above result, if the gauge class of a flat connection is equivariant with respect to the hyperelliptic involution ψ and with respect to ρ , it is also equivariant with respect to τ . The converse is true as well:

Proposition 22. *Let ∇ be a flat, irreducible, τ -equivariant $SL(2, \mathbb{C})$ -connection on a Riemann surface Σ of genus 3, where τ is a fix-point free involution. Then, ∇ is also equivariant with respect to the hyperelliptic involution.*

Proof. As ∇ is equivariant with respect to τ it is given by the pull-back of a flat connection on the genus 2 surface $\Sigma_\tau = \Sigma/\tau$. If ∇ is irreducible, the corresponding connection on Σ_τ is irreducible as well. Hence, by a result of [13], the connection is equivariant with respect to the hyperelliptic involution on the genus 2 surface, and hence also on the genus 3 surface. \square

Remark 23. *Let Γ be the group generated by ψ and ρ . The proposition above states that (stable) Γ -equivariant Higgs bundles are exactly (stable) τ -invariant Higgs bundles. Hence, the intersection of the space of equivalence classes of Γ -equivariant Higgs bundles with a regular fibre of the Hitchin map is given by Theorem 13, see also Theorem 24 below.*

Note that irreducibility is not a necessary condition to be in the equivariant brane, as there exist flat reducible connection on Σ which correspond to irreducible connections on the hyperelliptic genus 2 surface Σ_τ . Furthermore, since flat abelian connections on the quotient Σ_τ are equivariant with respect to the hyperelliptic involution, a non-trivial class in $H^1(\Sigma_\tau, \mathbb{Z}_2)$ is given by the choice of two points (w_1, w_2) out of six Weierstrass-points of Σ_τ . Such a class is represented by a closed curve on $\mathbb{CP}^1 \setminus \{w_1, \dots, w_6\}$ with even winding number around w_3, \dots, w_6 and odd winding number around w_1 and w_2 . Without loss of generality, suppose that Σ_τ is given by $y^2 = (z - z_1)(z - z_2)\dots(z - z_6)$ and that the class in $H^1(\Sigma_\tau, \mathbb{Z}_2)$ labelling the double cover $\pi: \Sigma \rightarrow \Sigma_\tau$ is determined by $w_1 = z_1$ and $w_2 = z_2$. Then, the Riemann surface Σ has equation $u^2 = (z_1 - z_2)^2(z_2 - z_3)\dots(z_2 - z_6)(w^2 - \frac{z_1 - z_3}{z_2 - z_3})\dots(w^2 - \frac{z_1 - z_6}{z_2 - z_6})$, the fix point free involution τ is given by $(u, w) \mapsto (-u, -w)$, and the covering map $\pi: \Sigma \rightarrow \Sigma_\tau$ is

$$(24) \quad \pi: (u, w) \mapsto (y, z) = \left(\frac{uw}{(w^2 - 1)^3}, \frac{z_2 w^2 - z_1}{w^2 - 1} \right).$$

In order to study the $SL(2, \mathbb{C})$ Hitchin fibration on Σ , and the one induced on Σ_τ , consider Q a holomorphic quadratic differential on Σ_τ with simple zeros. After a Moebius transformation, and up to constant scaling $Q = \frac{z(dz)^2}{y^2}$, and its pull-back to Σ is $\pi^*Q = 4 \frac{(z_1 - z_2)^2(w^2 - 1)(z_2 w^2 - z_1)(dw)^2}{u^2}$. The quadratic differentials Q and π^*Q label $SL(2, \mathbb{C})$ -Higgs bundles, and thus define spectral curves S and S_τ which are double covers of Σ and Σ_τ , respectively. Thus, there is the following natural commutative diagram, where as in previous sections, σ is the natural involution switching the sheets of the 2-covers:

$$(25) \quad \begin{array}{ccccc} S & \xrightarrow{\quad \text{mod } \tau \quad} & S_\tau := S/\tau & \xrightarrow{\quad \text{mod } \psi \quad} & \tilde{\Sigma} := S_\tau/\psi \\ \downarrow \text{mod } \sigma & & \downarrow \text{mod } \sigma & & \downarrow \text{mod } \sigma \\ \Sigma := S/\sigma & \xrightarrow{\quad \text{mod } \tau \quad} & \Sigma_\tau := \Sigma/\tau & \xrightarrow{\quad \text{mod } \psi \quad} & \mathbb{CP}^1 \end{array}$$

Note that $\tilde{\Sigma} \rightarrow \mathbb{CP}^1$ branches over the points $0, \infty, z_1, \dots, z_6 \in \mathbb{CP}^1$. Denote the corresponding Weierstrass points of $\tilde{\Sigma}$ by the same symbols. Moreover, one can see that the unbranched cover

$S_\tau \rightarrow \tilde{\Sigma}$ corresponds to the pair of Weierstrass points $0, \infty \in \tilde{\Sigma}$, and that the unbranched cover $S_\psi \rightarrow \tilde{\Sigma}$ corresponds to the pair of Weierstrass points $z_1, z_2 \in \tilde{\Sigma}$, for $S_\psi := S/\psi$.

In order to describe the points in the regular Hitchin fibres corresponding to τ -anti-equivariant Higgs fields, recall that these are given by the Prym of $\tilde{S} = S/(\tau \circ \sigma)$. Consider the map $\tilde{S} \rightarrow \Sigma_\tau$ given by the spectral curve $S_\tau \rightarrow \Sigma_\tau$ twisted by the holomorphic \mathbb{Z}_2 bundle $L_0 = L(z_1 - z_2)$ on Σ_τ , where $L_0 \in \text{Jac}(\tilde{\Sigma})$ is a holomorphic line bundle whose pull-back to S is trivial. Moreover, one has that $S_\psi/(\tau \circ \sigma)$ is a Riemann surface of genus 2, and the natural action of σ on $S_\psi/(\tau \circ \sigma)$ has 6 fix points lying over $0, \infty, z_3, z_4, z_5, z_6$. Equivalently, $S_\psi \rightarrow S_\psi/(\tau \circ \sigma)$ branches over the 4 points lying over $z_1, z_2 \in \mathbb{CP}^1$, and the Riemann surface $E := (S_\psi/(\tau \circ \sigma))/(\sigma \circ \psi)$ is of genus 1 and branches over $0, \infty, z_1, z_2 \in \mathbb{CP}^1$. Hence, one has that:

Theorem 24. *The (B, B, B) -brane of Γ -equivariant $SL(2, \mathbb{C})$ -Higgs bundles intersects the generic fibres of the Hitchin fibration in an abelian variety*

$$(26) \quad \mathcal{P} = \text{Jac}(\tilde{\Sigma})/\mathbb{Z}_2 \times \mathbb{Z}_2,$$

where $\tilde{\Sigma}$ is the hyperelliptic Riemann surface of genus 3 branched over $0, \infty, z_1, \dots, z_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to the group generated by the \mathbb{Z}_2 -bundles $L_0(0 - \infty)$ and $L_0(z_1 - z_2)$. Equivalently, one has that

$$(27) \quad \mathcal{P}^\vee = \text{Jac}(E) \times \text{Jac}(\tilde{S}_\tau)/\mathbb{Z}_2 \times \mathbb{Z}_2,$$

where $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to the group generated by the \mathbb{Z}_2 -bundles $L_0(0 - \infty)$ and $L_0(z_1 - z_2)$.

Proof. It only remains to prove the second part of the theorem. In order to do so, note that the natural map (pull-back composed with tensor product)

$$\text{Jac}(E) \times \text{Jac}(\tilde{\Sigma}_\tau) \rightarrow \text{Prym}(S \rightarrow \Sigma)$$

is surjective, and its kernel is of course finite. Note that any non-trivial line bundle on \tilde{S}_τ which pulls back to the trivial bundle on S is already the specific \mathbb{Z}_2 corresponding to the unbranched double covering $S \rightarrow \tilde{S}_\tau$. This \mathbb{Z}_2 -bundle is given by $L(z_1^+ + z_1^- - z_2^+ - z_2^-) \rightarrow \tilde{S}_\tau$, which itself is the pull back of $L_0(z_1 - z_2) \rightarrow E$. Thus (by tensoring with this bundle if necessary) one can restrict to the case of the trivial bundle $\underline{\mathbb{C}} \rightarrow \tilde{S}_\tau$, and ask which holomorphic bundles $l_1 \rightarrow E$ and $l_2 \rightarrow \tilde{\Sigma}_\tau$ have the property that product of their pull-back to \tilde{S}_τ is trivial. Note that bundles in the pull-back of $\text{Jac}(E)$ are ψ -anti-invariant while those in the pull-back of $\text{Jac}(\tilde{\Sigma}_\tau)$ are ψ -invariant. Hence the intersection of the pull-back of the two Jacobians is of dimension 0 and consists of \mathbb{Z}_2 -bundles on \tilde{S}_τ . Therefore they also must be \mathbb{Z}_2 -bundles on E and $\tilde{\Sigma}_\tau$. From this one has that $l_1 = L_0(0 - \infty) \rightarrow E$ and $l_2 = L_0(0 - \infty) \rightarrow \tilde{\Sigma}_\tau$ or both must be trivial. \square

Remark 25. *For Γ the group generated by ψ and ρ , note that Γ -anti-equivariant Higgs fields have not been defined. To see why this was not done, note that the space $\{L \in \text{Prym}(S, \Sigma) \mid \psi^*L = L^*, \rho^*L = L^*\}$, the space $\{L \in \text{Prym}(S, \Sigma) \mid \psi^*L = L^*, \rho^*L = L\}$ and the space $\{L \in \text{Prym}(S, \Sigma) \mid \psi^*L = L, \rho^*L = L^*\}$ are 0, 2 and 1 dimensional, respectively. Hence, the only reasonable way to get a half-dimensional space would be to consider the space of Higgs bundles for which there exists a $g \in \Gamma$ with respect to which the Higgs field is anti-invariant, and this is the space of τ -anti-equivariant Higgs bundles.*

7. SOME REMARKS ON EQUIVARIANT BRANES AND LANGLANDS DUALITY

Langlands duality can be seen in terms of Higgs bundles as a duality between the fibres of the Hitchin fibrations for $\mathcal{M}_{G_{\mathbb{C}}}$ and $\mathcal{M}_{L_{G_{\mathbb{C}}}}$, for ${}^L G_{\mathbb{C}}$ the Langlands dual group of $G_{\mathbb{C}}$ (as was first seen in [12]). As explained [18, Section 12] under the duality, specifically homological mirror symmetry, there should be an equivalence of categories of branes on $\mathcal{M}_G(\Sigma)$ and $\mathcal{M}_{L_G}(\Sigma)$ under which the brane types $(B, B, B) \leftrightarrow (B, A, A)$ are exchanged.

Examples of branes and their proposed duals in the moduli spaces of Higgs bundles for low rank groups were presented in [18], and further studied in [11]. Moreover, in the case of the (B, A, A) -branes coming from real forms G of the complex lie group $G_{\mathbb{C}}$ there is a conjecture of what the (support of) the dual branes should look like [5] (see [9] and [17] for support). It is thus natural to ask equivalent questions in the setting of the present research, about what the duality between branes should be in the case of the spaces constructed in this paper. In the case of $U(m, m)$ -Higgs bundles, the duality was studied in [17] where through the spectral data description of [27] in terms of anti-invariant line bundles, the proposed dual branes were constructed in terms of invariant ones, which agreed with the conjecture in [5].

In [17] Hitchin proposed a hyper-holomorphic sheaf which together with the hyper-Kähler subspace of the moduli space of Higgs bundles would give the (B, B, B) -brane. It is interesting to note that given the similarities of the construction of the subspaces of Higgs bundles in terms of equivariant objects, the hyperholomorphic sheaf constructed in [17] for $U(m, m)$ -Higgs bundles should give naturally a hyperholomorphic sheaf for the equivariant (B, B, B) -branes of the present paper.

Since from the work of Section 5.2 and the previous propositions, the branes obtained are subspaces of abelian varieties too, following the lines of thought of the real case one may think that the mirror of the equivariant (B, B, B) -brane is given by moduli space of Higgs bundles on Σ which are ψ -equivariant and anti-equivariant with respect to τ . Here, anti-equivariant means that the corresponding holomorphic structure $\bar{\partial}$ is equivariant, i.e., $\tau^*\bar{\partial} = \bar{\partial}.g$ and $\tau^*\Phi = -g^{-1}\Phi g$ for a suitable gauge transformation g . Note that the determinant of an anti-equivariant Higgs field is an invariant holomorphic quadratic differential Q . As before, after the choice of a square root of Q , anti-equivariant Higgs fields with determinant Q are parametrized by the abelian variety \mathcal{P}^\vee which consists of points in the Prym(S, Σ) which are invariant under ψ and anti-invariant under τ .

REFERENCES

- [1] J.E. Andersen, J. Grove, *Automorphism fixed points in the moduli space of semi-stable bundles*, Quart. J. Math. 57 (2006).
- [2] H. Babu, P. Venkataraman, *Group action on genus 3 curves and their Weierstrass points*, Computational Aspects of Algebraic Curves, in: Lecture Notes Ser. Comput., vol. 13, World Sci. Publ., Hackensack, NJ.
- [3] D. Baraglia, L.P. Schaposnik, *Monodromy of rank 2 twisted Hitchin systems and real character varieties*, Arxiv. 1506.00372.
- [4] D. Baraglia, L.P. Schaposnik, *Higgs bundles and (A, B, A) -branes*, Commun.Math.Phys. 331 (2014).
- [5] D. Baraglia, L.P. Schaposnik, *Real structures on moduli spaces of Higgs bundles*, [arXiv:1309.1195](https://arxiv.org/abs/1309.1195). to appear in Advances in Theoretical and Mathematical Physics (2015).
- [6] S.A. Broughton, *Classifying finite group actions on surfaces of low genus*, Journal of Pure and Applied Algebra **69** 3 (1991), p. 233-270.
- [7] E. Bujalance, F.J. Cirrea, M.D.E. Conderb, B. Szepietowski, *Finite group actions on bordered surfaces of small genus* Journal of Pure and Applied Algebra **214** 12 (2010), p. 2165-2185
- [8] M. Braverman, M. Farber, *Novikov inequalities with symmetry*, C. R. Acad. Sci. Paris **323** (1996).
- [9] D. Gaiotto, *S-duality of boundary conditions and the Geometric Langlands program*, [arXiv:1609.09030](https://arxiv.org/abs/1609.09030)
- [10] B. H. Gross, J. Harris, *Real algebraic curves*, Ann. Sci. École Norm. Sup. (4) **14** (1981), no.2, 157-182.
- [11] S. Gukov, *Quantization via Mirror Symmetry*, Jpn. J. Math. (2011) 6: 65.
- [12] T. Hausel, M. Thaddeus, *Mirror Symmetry, Langlands Duality, and the Hitchin System*, Invent. math., 153 1, (2003) 197-229.
- [13] V. Heu, F. Loray *Flat rank 2 vector bundles on genus 2 curves*, [arXiv:1401.2449](https://arxiv.org/abs/1401.2449), (2015)
- [14] N.J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. **55** 3, (1987).
- [15] N.J. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. 54 1 (1987) 91-114.
- [16] N.J. Hitchin, *Lie Groups and Teichmüller Space*, Topology **31** 3, (1992), 449-473.
- [17] N.J. Hitchin, *Higgs bundles and characteristic classes*, [arXiv:1308.4603](https://arxiv.org/abs/1308.4603) (2013).
- [18] A. Kapustin, E. Witten, *Electric-magnetic duality and the geometric Langlands program*. Commun. Number Theory Phys. **1**, (2007) 1-236.
- [19] H. Konno, *Construction of the moduli space of stable parabolic Higgs bundles on a Riemann surface*. J. Math. Soc. Jap. Vol. 45, No. 2, 1993.

- [20] K. Magaard, T. Shaska, S. Shpectorov and H. Völklein , *The locus of curves with prescribed automorphism group*, RIMS Publication series 1267 (2002) 112-141.
- [21] R. Miranda, *Algebraic curves and Riemann surfaces*
- [22] D. Nadler, *Perverse sheaves on real loop Grassmannians*, Invent. math, 159 1, (2005) 1-73.
- [23] H. Nakajima, *Hyper-Kähler structures on moduli spaces of parabolic Higgs bundles on Riemann surfaces*, Moduli of vector bundles (Sanda, 1994; Kyoto, 1994), 199–208, Lec. Notes in Pure and Appl. Math., 179.
- [24] B. Nasatyr, B. Steer, *Orbifold Riemann surfaces and the Yang-Mills-Higgs equations*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **22** 4 (1995), p. 595-643
- [25] L.P. Schaposnik, *Spectral data for G -Higgs bundles*, University of Oxford DPhil Thesis, [arXiv:1301.1981](https://arxiv.org/abs/1301.1981).
- [26] L. P. Schaposnik, *Monodromy of the SL_2 Hitchin fibration*. Internat. J. Math. **24** (2013), no. 2.
- [27] L.P. Schaposnik, *Spectral data for $U(m,m)$ -Higgs bundles*, Int Math Res Notices, 2015 11, 3486-3498. DOI: 10.1093/imrn/rnu029 (2015).
- [28] C. Simpson, *Higgs bundles and local systems*, Inst.Hautes Études Sci.Publ.Math. 75 (1992), 5-95

SEBASTIAN HELLER - MATHEMATISCHES INSTITUT, EBERHARD KARLS UNIVERSITÄT TÜBINGEN, 72074 TÜBINGEN, GERMANY.

E-mail address: heller@mathematik.uni-tuebingen.de

LAURA P. SCHAPOSNIK - DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, CHICAGO, 60647, USA

E-mail address: schapos@uic.edu